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Abstract
The generalized sine-Gordon (sG) equation was derived as an integrable
generalization of the sG equation. In this paper, we develop a direct method for
solving the generalized sG equation without recourse to the inverse scattering
method. In particular, we construct multisoliton solutions in the form of
parametric representation. We obtain a variety of solutions which include kinks,
loop solitons and breathers. The properties of these solutions are investigated
in detail. We find a novel type of solitons with a peculiar structure that the
smaller soliton travels faster than the larger soliton. We also show that the
short-pulse equation describing the propagation of ultra-short pulses is reduced
from the generalized sG equation in an appropriate scaling limit. Subsequently,
the reduction to the sG equation is briefly discussed.

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1. Introduction

We consider the following generalized sine-Gordon (sG) equation:

$$u_{tx} = \left(1 + \nu \partial_x^2\right) \sin u, \tag{1.1}$$

where $u = u(x, t)$ is a scalar-valued function, ν is a real parameter, $\partial_x^2 = \partial^2/\partial x^2$ and the
subscripts t and x appended to u denote partial differentiation. The generalized sG equation has
been derived for the first time in [1] using bi-Hamiltonian methods. Quite recently, equation
(1.1) with $\nu < 0$ was shown to be a completely integrable partial differential equation (PDE)
[2]. Indeed, constructing a Lax pair associated with it, the initial value problem of the
equation was solved for decaying initial data. In the process, the Riemann–Hilbert formalism
was developed to obtain eigenfunctions of the Lax pair. Soliton solutions are obtainable in
principle, but their derivation needs a very complicated procedure. Although some qualitative

features of traveling-wave solutions are discussed in a different context from the Riemann–Hilbert formalism, explicit expressions of solutions are not available as yet.

The purpose of this paper is to obtain exact solutions of equation (1.1) with $\nu < 0$ and discuss their properties. We consider real and nonperiodic solutions. In our analysis, we take $\nu = -1$ without loss of generality. The exact method of solution used here is the bilinear transformation method which is a very powerful tool in obtaining special solutions of soliton equations [3, 4]. The method has wide applications ranging from continuous to discrete soliton equations. The central problem in the bilinear formalism is the construction of tau functions which are introduced through dependent variable transformations.

This paper is organized as follows. In section 2, we develop an exact method of solution. Specifically, we use a hodograph transformation to transform the equation under consideration into a more tractable form. The transformed equation is further put into a system of bilinear equations by introducing appropriate dependent variable transformations. We then construct explicit solutions of the bilinear equations by means of a standard procedure in the bilinear formalism. The multisoliton solutions are obtained in the form of parametric representation. In section 3, we describe properties of solutions. First, we consider 1-soliton solutions which include kink and loop soliton solutions as well as a new type of multivalued functions. Throughout this paper, we use the term 'soliton' as a generic name of elementary solutions such as kink, loop soliton and breather solutions. A novel feature of regular kink solutions is found which has never been seen in the sG kinks. Next, the 2-soliton solutions are discussed. We address the kink–kink and kink–loop soliton solutions together with the 1-breather solution. Last, we explore the general multisoliton solutions. Our particular concern is the multikink solution for which the large-time asymptotic is derived and the associated formula for the phase shift is obtained. A recipe for constructing the multibreather solutions is briefly described. As examples of the multisoliton solutions, we present a solution describing the interaction between a soliton and a breather as well as a 2-breather solution which are reduced from the 3- and 4-soliton solutions, respectively. In section 4, we point out a close relationship between the generalized sG equation and the short-pulse equation which models the propagation of ultra-short optical pulses. We show that the generalized sG equation is reduced to the short-pulse equation by taking an appropriate scaling limit. The parametric multiloop soliton solution of the short-pulse equation presented in [5] is reproduced from the corresponding one for the generalized sG equation. The similar limiting procedure is also applied to the formula for the phase shift. Subsequently, the reduction of the generalized sG equation to the sG equation is discussed shortly. Section 5 is devoted to conclusions. In appendix A, we show that the tau functions for the multisoliton solutions obtained in section 3 satisfy a system of bilinear equations. The proof is carried out by means of an elementary method using various formulas for determinants. In appendix B, we derive the 1-soliton solutions by an alternative method and demonstrate that they reproduce the corresponding 1-soliton solutions obtained in section 3.

2. Exact method of solution

2.1. Hodograph transformation

We introduce the new dependent variable r in accordance with the relation

$$r^2 = 1 + u_x^2, \qquad (r > 0), \tag{2.1}$$

to transform equation (1.1) with $\nu = -1$ into the form

$$r_t + (r \cos u)_x = 0. \tag{2.2}$$

We then define the hodograph transformation $(x, t) \rightarrow (y, \tau)$ by

$$\mathrm{d}y = r\,\mathrm{d}x - r\cos u\,\mathrm{d}t, \qquad \mathrm{d}\tau = \mathrm{d}t. \tag{2.3}$$

The x and t derivatives are then rewritten in terms of the y and τ derivatives as

$$\frac{\partial}{\partial x} = r\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - r\cos u\frac{\partial}{\partial y}. \tag{2.4}$$

With the new variables y and τ, (2.1) and (2.2) are recast into the form

$$r^2 = 1 + r^2 u_y^2, \tag{2.5}$$

$$\left(\frac{1}{r}\right)_\tau - (\cos u)_y = 0, \tag{2.6}$$

respectively. Further reduction is possible if one defines the variable ϕ by

$$u_y = \sin\phi, \qquad \phi = \phi(y, \tau) \quad \left(-\frac{\pi}{2} < \phi < \frac{\pi}{2}, \mathrm{mod}\, 2\pi\right). \tag{2.7}$$

It follows from (2.5) and (2.7) that

$$\frac{1}{r} = \cos\phi. \tag{2.8}$$

Substituting (2.7) and (2.8) into equation (2.6), we find

$$\phi_\tau = \sin u. \tag{2.9}$$

If we eliminate the variable ϕ from (2.7) and (2.9), we obtain a single PDE for u

$$\frac{u_{\tau y}}{\sqrt{1 - u_y^2}} = \sin u. \tag{2.10}$$

Similarly, elimination of the variable u gives a single PDE for ϕ :

$$\frac{\phi_{\tau y}}{\sqrt{1 - \phi_\tau^2}} = \sin\phi. \tag{2.11}$$

By inverting the relationship (2.4) and using (2.8), the equation that determines the inverse mapping $(y, \tau) \rightarrow (x, t)$ is found to be governed by the system of linear PDEs for $x = x(y, \tau)$

$$x_y = \cos\phi, \tag{2.12a}$$

$$x_\tau = \cos u. \tag{2.12b}$$

Note that the integrability of the system of equations (2.12) is assured by (2.7) and (2.9).

A sequence of transformations described above are almost the same as those employed for solving the short-pulse equation [5]. The underlying idea is to transform the original equation to the (possibly) integrable equation. In the case of the short-pulse equation, the transformed equation is the sG equation whereas in the present case, the corresponding equations are (2.10) and (2.11). The soliton solutions of the latter equations will be constructed here for the first time. Given u and ϕ, the most difficult problem is how to integrate the system of equations (2.12). This becomes the core part of the present paper and will be resolved by theorem 2.1.

2.2. Bilinear formalism

Here, we develop a method for solving a system of PDEs (2.7) and (2.9). We use the bilinear transformation method [3, 4]. Let σ and σ' be solutions of the sG equation:

$$\sigma_{\tau y} = \sin \sigma, \qquad \sigma = \sigma(y, \tau), \tag{2.13a}$$

$$\sigma'_{\tau y} = \sin \sigma', \qquad \sigma' = \sigma'(y, \tau). \tag{2.13b}$$

We then put

$$u = \tfrac{1}{2}(\sigma + \sigma'), \tag{2.14a}$$

$$\phi = \tfrac{1}{2}(\sigma - \sigma'). \tag{2.14b}$$

In terms of σ and σ', equations (2.7) and (2.9) can be written as

$$\tfrac{1}{2}(\sigma + \sigma')_y = \sin \tfrac{1}{2}(\sigma - \sigma'), \tag{2.15a}$$

$$\tfrac{1}{2}(\sigma - \sigma')_\tau = \sin \tfrac{1}{2}(\sigma + \sigma'). \tag{2.15b}$$

It should be remarked that the system of PDEs (2.15) constitutes a Bäcklund transformation of the sG equation with a Bäcklund parameter taken to be 1 [6]. The real-valued solutions of the sG equations (2.13) can be put into the form [7–10]

$$\sigma = 2\mathrm{i} \ln \frac{f^*}{f}, \tag{2.16a}$$

$$\sigma' = 2\mathrm{i} \ln \frac{g^*}{g}, \tag{2.16b}$$

where f^* and g^* denote the complex conjugate of f and g, respectively. The tau functions f and g play a central role in our analysis. They are fundamental quantities in constructing solutions. For soliton solutions, they satisfy the following bilinear equations [7]:

$$D_\tau D_y f \cdot f = \tfrac{1}{2}(f^2 - f^{*2}), \tag{2.17a}$$

$$D_\tau D_y g \cdot g = \tfrac{1}{2}(g^2 - g^{*2}), \tag{2.17b}$$

where the bilinear operators D_τ and D_y are defined by

$$D_\tau^m D_y^n f \cdot g = (\partial_\tau - \partial_{\tau'})^m (\partial_y - \partial_{y'})^n f(\tau, y) g(\tau', y')|_{\tau'=\tau, y'=y} \quad (m, n = 0, 1, 2, \ldots). \tag{2.18}$$

Now, we seek solutions of equations (2.7) and (2.9) of the form

$$u = \mathrm{i} \ln \frac{F^*}{F}, \tag{2.19a}$$

$$\phi = \mathrm{i} \ln \frac{G^*}{G}, \tag{2.19b}$$

where F and G are new tau functions. It turns out from (2.14), (2.16) and (2.19) that

$$2u = \sigma + \sigma' = 2\mathrm{i} \ln \frac{f^* g^*}{f g} = 2\mathrm{i} \ln \frac{F^*}{F}, \tag{2.20a}$$

4

$$2\phi = \sigma - \sigma' = 2\mathrm{i}\ln\frac{f^*g}{fg^*} = 2\mathrm{i}\ln\frac{G^*}{G}. \tag{2.20b}$$

By taking into account (2.20), we may assume the following relations among the tau functions f, g, F and G:

$$F = fg, \tag{2.21a}$$

$$G = fg^*. \tag{2.21b}$$

The above expressions lead to an important relation

$$F^*F = G^*G. \tag{2.22}$$

Substituting (2.19) into equations (2.7) and (2.9) and using (2.22), we see that F and G satisfy a system of bilinear equations

$$D_y F^* \cdot F = -\tfrac{1}{2}(G^2 - G^{*2}), \tag{2.23a}$$

$$D_\tau G^* \cdot G = -\tfrac{1}{2}(F^2 - F^{*2}). \tag{2.23b}$$

Thus, the problem under consideration is reduced to obtain solutions of equations (2.23) subjected to condition (2.22). After some trials, however, we found that this procedure for constructing solutions is difficult to perform. Hence, we employ an alternative approach. To begin with, we impose the following bilinear equations for f and g which turn out to be the starting point in our analysis:

$$D_y f \cdot g^* = \tfrac{1}{2}(fg^* - f^*g), \tag{2.24a}$$

$$D_\tau f \cdot g = \tfrac{1}{2}(fg - f^*g^*). \tag{2.24b}$$

With (2.24) at hand, the following proposition holds.

Proposition 2.1. *If f and g satisfy the bilinear equations (2.24), then the tau functions F and G defined by (2.21) satisfy the bilinear equations (2.23).*

Proof. First, we prove (2.23a). We substitute (2.21a) into the left-hand side of (2.23a) and rewrite it in terms of the bilinear operator to obtain

$$D_y F^* \cdot F = -(D_y f \cdot g^*)f^*g + (D_y f^* \cdot g)fg^*. \tag{2.25}$$

By virtue of (2.24a), the right-hand side of (2.25) becomes $-(1/2)\{(fg^*)^2 - (f^*g)^2\}$ which is equal to the right-hand side of (2.23a) by (2.21b). The proof of (2.23b) can be done in the same way by using (2.24b). $\qquad\square$

2.3. Parametric representation

We demonstrate that the solution of equation (1.1) with $\nu = -1$ admits a parametric representation. The following relation is crucial to integrate (2.12).

Proposition 2.2. $\cos\phi$ *is expressed in terms of f and g as*

$$\cos\phi = 1 + \left(\ln\frac{g^*g}{f^*f}\right)_y. \tag{2.26}$$

Proof. Using (2.24a), one obtains

$$\left(\ln \frac{g^* g}{f^* f} \right)_y = -\frac{D_y f \cdot g^*}{f g^*} - \frac{D_y f^* \cdot g}{f^* g}$$

$$= \frac{1}{2} \frac{(f g^*)^2 + (f^* g)^2}{f^* f g^* g} - 1. \tag{2.27}$$

On the other hand, it follows from (2.19b) and (2.21b) that

$$\cos \phi = \frac{1}{2} \left(\frac{G}{G^*} + \frac{G^*}{G} \right)$$

$$= \frac{1}{2} \frac{(f g^*)^2 + (f^* g)^2}{f^* f g^* g}. \tag{2.28}$$

Relation (2.26) follows immediately by comparing (2.27) and (2.28). □

Integrating (2.12a) coupled with (2.26) by y, we obtain the expression of x

$$x = y + \ln \frac{g^* g}{f^* f} + d(\tau), \tag{2.29}$$

where d is an integration constant which depends generally on τ. Expression (2.29) now leads to our main result.

Theorem 2.1. *The real-valued solution of equation (1.1) with $\nu = -1$ can be written by the parametric representation*

$$u(y, \tau) = i \ln \frac{f^* g^*}{f g}, \tag{2.30a}$$

$$x(y, \tau) = y + \tau + \ln \frac{g^* g}{f^* f} + y_0, \tag{2.30b}$$

where the tau functions f and g satisfy both (2.17) and (2.24) simultaneously and y_0 is an arbitrary constant independent of y and τ.

Proof. Expression (2.30a) for u is a consequence of (2.19a) and (2.21a). To prove (2.30b), we substitute (2.29) into (2.12b) and obtain the relation

$$\cos u = \left(\ln \frac{g^* g}{f^* f} \right)_\tau + d'(\tau). \tag{2.31}$$

The left-hand side of (2.31) can be expressed by f and g in view of (2.30a) whereas the right-hand side is modified by using (2.24b). After a few calculations, we find that most terms are cancelled, leaving the equation $d'(\tau) = 1$. Integrating this equation, one obtains $d(\tau) = \tau + y_0$, which, substituted into (2.29), gives expression (2.30b) for x. □

The parametric solution (2.30) would produce in general a multi-valued function as happened in the case of the short-pulse equation [5]. To derive a criterion for single-valued functions, we calculate u_x with use of (2.7) and (2.8) and obtain

$$u_x = r u_y = \tan \phi. \tag{2.32}$$

Thus, if the inequality $-\pi/2 < \phi < \pi/2$ (mod π) holds for all y and τ, then u becomes a regular function of x and t. By virtue of the identity

$$i \ln \frac{G^*}{G} = 2 \tan^{-1} \left(\frac{\text{Im} G}{\text{Re} G} \right), \tag{2.33}$$

as well as the relation (2.21*b*), the above condition for regularity can be written as

$$-1 < \frac{\text{Im}(fg^*)}{\text{Re}(fg^*)} < 1. \tag{2.34}$$

In the case of 1-soliton solutions discussed in the next section, the above condition is found explicitly in terms of the parameters characterizing solutions.

2.4. Multisoliton solutions

The last step in constructing solutions is to find the tau functions f and g for the sG equation which satisfy simultaneously the bilinear equations (2.24). The following theorem establishes this purpose.

Theorem 2.2. *The tau functions f and g given below satisfy both the bilinear forms (2.17) of the gG equation and the bilinear equations (2.24)*

$$f = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\xi_j + \frac{\pi}{2}\,\text{i}\right) + \sum_{1 \leqslant j < k \leqslant N} \mu_j \mu_k \gamma_{jk}\right], \tag{2.35a}$$

$$g = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\xi_j - 2d_j + \frac{\pi}{2}\,\text{i}\right) + \sum_{1 \leqslant j < k \leqslant N} \mu_j \mu_k \gamma_{jk}\right], \tag{2.35b}$$

where

$$\xi_j = p_j y + \frac{1}{p_j}\tau + \xi_{j0} \qquad (j = 1, 2, \ldots, N), \tag{2.36a}$$

$$\text{e}^{\gamma_{jk}} = \left(\frac{p_j - p_k}{p_j + p_k}\right)^2 \qquad (j, k = 1, 2, \ldots, N;\ j \neq k), \tag{2.36b}$$

$$\text{e}^{-2d_j} = \frac{1 - p_j}{1 + p_j} \qquad (j = 1, 2, \ldots, N). \tag{2.36c}$$

Here, p_j and ξ_{j0} are arbitrary real parameters satisfying the conditions $p_j \neq p_k$ for $j \neq k$ and N is an arbitrary positive integer. The notation $\sum_{\mu=0,1}$ implies the summation over all possible combination of $\mu_1 = 0, 1, \mu_2 = 0, 1, \ldots, \mu_N = 0, 1$.

Proof. It has been shown that f and g given by (2.35*a*) and (2.35*b*) satisfy the bilinear equations (2.17*a*) and (2.17*b*), respectively [7]. Thus, it is sufficient to prove that they satisfy the bilinear equations (2.24). The proof is carried out by a lengthy calculation using various formulas for determinants. It will be summarized in appendix A. □

2.5. Remark

The tau functions f and g given by (2.35) yield real-valued solutions since all the parameters p_j and ξ_{j0} ($j = 1, 2, \ldots, N$) are chosen to be real numbers. If one looks for breather solutions, for example, one needs to introduce complex parameters (see sections 3.2.3, 3.3.2 and 3.3.3). Even in this case, however, the analysis developed here can be applied as well without making essential changes. Actually, we may use the tau functions f' and g' instead of f^* and g^*, respectively, where f' and g' are obtained simply from f and g by replacing i by $-$i, but all

the parameters in the tau functions are assumed to be complex numbers. The solutions of the sG equations (2.13a) and (2.13b) can be written as $\sigma = 2\mathrm{i} \ln(f'/f)$ and $\sigma' = 2\mathrm{i} \ln(g'/g)$, respectively, where the tau functions f, f', g and g' satisfy the following systems of bilinear equations:

$$D_\tau D_y f \cdot f = \tfrac{1}{2}(f^2 - f'^2), \qquad D_\tau D_y f' \cdot f' = \tfrac{1}{2}(f'^2 - f^2), \qquad (2.37a)$$

$$D_\tau D_y g \cdot g = \tfrac{1}{2}(g^2 - g'^2), \qquad D_\tau D_y g' \cdot g' = \tfrac{1}{2}(g'^2 - g^2). \qquad (2.37b)$$

The bilinear equations corresponding to (2.24) are then given by

$$D_y f \cdot g' = \tfrac{1}{2}(fg' - f'g), \qquad D_\tau f \cdot g = \tfrac{1}{2}(fg - f'g'), \qquad (2.38a)$$

$$D_y f' \cdot g = \tfrac{1}{2}(f'g - fg'), \qquad D_\tau f' \cdot g' = \tfrac{1}{2}(f'g' - fg). \qquad (2.38b)$$

The expressions corresponding to (2.19)–(2.23) are obtained if one replaces the asterisk appended to the tau functions by the prime. Under these modifications, the real-valued solutions are produced if one imposes the conditions $f' = f^*$ and $g' = g^*$. See also an analogous problem associated with breather solutions of the short-pulse equation [5].

3. Properties of solutions

The parametric representation (2.30) of the solution with the tau functions given by (2.35) exhibit a variety of soliton solutions of equation (1.1) with $\nu = -1$. As exemplified here, solutions include both single-valued and multi-valued kinks, loop solitons and breathers.

3.1. 1-soliton solutions

The tau functions for the 1-soliton solutions are given by (2.35) with $N = 1$:

$$f = 1 + \mathrm{i}\,\mathrm{e}^{\xi_1}, \qquad \xi_1 = p_1 y + \frac{\tau}{p_1} + \xi_{10}, \qquad (3.1a)$$

$$g = 1 + \mathrm{i} s_1\,\mathrm{e}^{\xi_1}, \qquad s_1 = \frac{1 - p_1}{1 + p_1}. \qquad (3.1b)$$

The parameters p_1 and ξ_{10} are related to the amplitude and phase of the soliton, respectively and ξ_1 is a phase variable characterizing the parametric representation of the solution. It follows from (2.21) and (3.1) that

$$F = 1 - s_1\,\mathrm{e}^{2\xi_1} + \mathrm{i}(1 + s_1)\,\mathrm{e}^{\xi_1}, \qquad (3.2a)$$

$$G = 1 + s_1\,\mathrm{e}^{2\xi_1} + \mathrm{i}(1 - s_1)\,\mathrm{e}^{\xi_1}. \qquad (3.2b)$$

Using (2.30) and (3.1), the parametric representation of the solution is written in the form

$$u = 2\tan^{-1}(\sinh \xi_1 - p_1 \cosh \xi_1) + \pi, \qquad (3.3a)$$

$$x = y + \tau + \ln\left(\frac{1 + p_1^2}{(1 + p_1)^2} - \frac{2p_1}{(1 + p_1)^2}\tanh \xi_1\right) + y_0, \qquad (3.3b)$$

where we have imposed the boundary condition $u(-\infty, t) = 0$. Note that if u solves equation (1.1), then so do the functions $\pm u + 2\pi n$ (n : integer). To describe solutions

of traveling-wave type like 1-soliton solutions, it is convenient to parameterize the solutions in terms of the single variable ξ_1. For this purpose, we rewrite (3.3b) as

$$X \equiv x + c_1 t + x_0 = \frac{\xi_1}{p_1} + \ln\left(\frac{1+p_1^2}{(1+p_1)^2} - \frac{2p_1}{(1+p_1)^2}\tanh\xi_1\right) + y_0, \qquad (3.4a)$$

where c_1 is the velocity of the soliton given by

$$c_1 = \frac{1}{p_1^2} - 1, \qquad (3.4b)$$

and $x_0 = \xi_{10}/p_1$. Observing the motion of the soliton in the original (x, t) coordinate system, it travels to the left at the constant velocity c_1 for $p_1 < 1$ and to the right for $p_1 > 1$. In the critical case $p_1 = 1$ for which $c_1 = 0$, the soliton remains stationary. The profile of the soliton changes drastically depending on values of the parameter p_1. The singular nature of the solution can be extracted conveniently from the information on the gradient of u with respect to X. A calculation using (3.3a) and (3.4) gives

$$u_X = \frac{1}{p_1 \cosh\xi_1} \frac{1 - p_1\tanh\xi_1}{\tanh^2\xi_1 - \frac{1}{p_1}\tanh\xi_1 + \frac{1-p_1^2}{2p_1^2}}. \qquad (3.5)$$

Let us now analyze solution (3.3). Using (3.1), condition (2.34) for single-valued function becomes

$$-1 < \frac{p_1}{\cosh\xi_1 - p_1\sinh\xi_1} < 1. \qquad (3.6)$$

In the following analysis, we assume $p_1 > 0$ without loss of generality. If $0 < p_1 < 1$, then (3.6) leads to the inequality

$$\tanh^2\xi_1 - \frac{1}{p_1}\tanh\xi_1 + \frac{1-p_1^2}{2p_1^2} > 0. \qquad (3.7)$$

One can see that inequality (3.7) always holds for $0 < p_1 < \frac{1}{\sqrt{2}}$. In this case, u_X from (3.5) becomes finite for arbitrary values of ξ_1. On the other hand, if $\frac{1}{\sqrt{2}} < p_1 < 1$, then the solution exhibits singularities at two points $X = X_\pm \equiv X(\xi_\pm)$ where

$$\xi_\pm = \tanh^{-1}\left[\frac{1}{2p_1}\left(1 \pm \sqrt{2p_1^2 - 1}\right)\right]. \qquad (3.8)$$

If $p_1 > 1$, inequality (3.6) breaks down, giving rise to two singular points whose positions are the same as $X_\pm$. Unlike the second case, however, u_X becomes zero at $X = X_0 \equiv X(\xi_0)$ with $\xi_0 = \tanh^{-1}(1/p_1)$, as readily noticed from (3.5). The solution for the case $p_1 = 1$ exhibits a peculiar behavior, which deserves a separate study. In view of these observations, the solutions can be classified to four types according to values of p_1, or equivalently c_1 by (3.4b), which we shall now investigate in detail.

Type 1. Regular kink: $c_1 > 1 \left(0 < p_1 < \frac{1}{\sqrt{2}}\right)$
Figure 1 shows a typical profile of u as a function of X. It exhibits a profile of a 2π-kink similar to the kink solution of the sG equation [6]. The propagation characteristic is, however, different from that of the sG kink. To clarify this point, we rewrite u_X from (3.5) as

$$u_X = \frac{2p_1}{\sqrt{1-p_1^2}} \frac{\cosh(\xi_1 - d_1)}{\cosh^2(\xi_1 - d_1) - \frac{p_1^2}{1-p_1^2}}, \qquad (3.9)$$
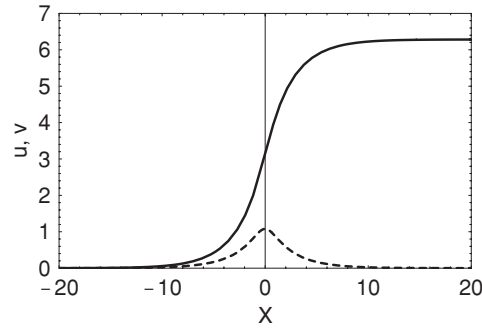
Figure 1. The profile of a regular kink u (solid line) and corresponding profile of $v \equiv u_X$ (broken line). The parameter p_1 is set to 0.4 and the parameter y_0 is chosen such that the center position of u_X is at $X = 0$.
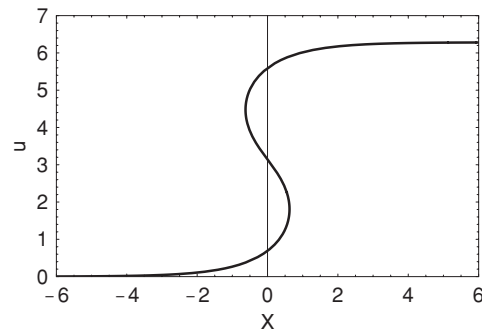
Figure 2. The profile of a singular kink with the parameter $p_1 = 0.9$.

where $d_1 = (1/2) \ln[(1 + p_1)/(1 - p_1)]$. It turns out from (3.9) that the amplitude A of u_X is given by

$$A = \frac{2p_1\sqrt{1 - p_1^2}}{1 - 2p_1^2}. \tag{3.10}$$

Eliminating p_1 from (3.4b) and (3.10), we find the dependence of the amplitude on the velocity

$$c_1 = \frac{1}{A^2}(A^2 + 2 + 2\sqrt{A^2 + 1}). \tag{3.11}$$

Relation (3.11) indicates that c_1 is a monotonically decreasing function of A. In other words, the smaller soliton travels faster than the larger soliton. This peculiar feature of the solution has never been observed in the behavior of the sG kink solutions. The broken line in figure 1 plots the profile of $v \equiv u_X$ obtained from the kink solution depicted in the same figure. It represents a soliton.

Type 2. Singular kink: $0 < c_1 < 1 \left(\frac{1}{\sqrt{2}} < p_1 < 1\right)$
The profile of the solution of type 2 is a 2π kink, but in the interval $X_- < X < X_+$, it becomes a three-valued function. Figure 2 shows a typical profile of u as a function of X.
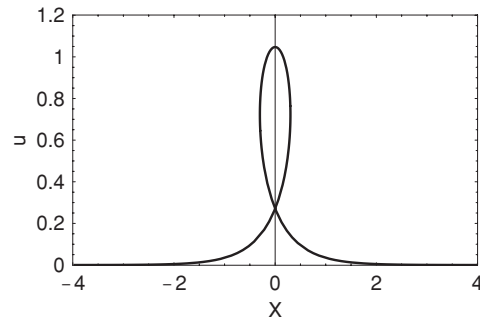
Figure 3. The profile of a loop soliton with the parameter $p_1 = 2.0$.
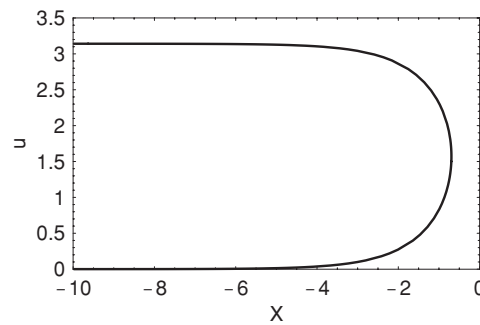

Figure 4. The profile of a stationary solution with the parameter $p_1 = 1.0$.

Type 3. Loop soliton: $-1 < c_1 < 0$ $(p_1 > 1)$

When compared with solutions of types 1 and 2, the solution for $p_1 > 1$ exhibits a different behavior. Indeed, we see from (3.3) and (3.5) that $u(\pm\infty) = 0$ and $u_X(X_0) = 0$, respectively. In addition, the solution has two singular points at $X = X_\pm$ such that $X_0 = (X_+ + X_-)/2$. Figure 3 shows a typical profile of u as a function of X. It represents a loop soliton and has a symmetrical profile with respect to a straight line $X = X_0$.

Type 4. Stationary solution: $c_1 = 1$ $(p_1 = 0)$

For this special value of the parameter p_1, the parametric solution (3.3a) and (3.4a) takes the form

$$u = -2 \tan^{-1} e^{-\xi_1} + \pi, \tag{3.12a}$$

$$X = -\ln(e^{\xi_1} + e^{-\xi_1}). \tag{3.12b}$$

Remarkably, one can eliminate the variable ξ_1 from this expression to give an explicit solution

$$\sin u = 2e^X, \qquad (-\infty < X < -\ln 2). \tag{3.13}$$

The profile of u is illustrated in figure 4 as a function of X. Since $c_1 = 0$, one has $X = x + x_0$ by (3.4a) and the solution becomes time-independent. Keeping this fact in mind, we can derive solution (3.13) directly from the stationary version of equation (1.1) with $\nu = -1$:

$$\left(1 - \frac{d^2}{dx^2}\right) \sin u = 0. \tag{3.14}$$

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Integrating equation (3.14) under the boundary condition $u(-\infty) = 0 \pmod{\pi}$, we recover (3.13).

The 1-soliton solutions presented above are of fundamental importance in constructing general multisoliton solutions. In fact, the latter solutions will be shown to consist of any combination of the former solutions. In appendix B, we shall derive the 1-soliton solutions by means of an elementary method.

3.2. 2-soliton solutions

The tau functions for the 2-soliton solutions read from (2.35) with $N = 2$ in the form

$$f = 1 + i(e^{\xi_1} + e^{\xi_2}) - \delta\, e^{\xi_1 + \xi_2}, \tag{3.15a}$$

$$g = 1 + i(s_1\, e^{\xi_1} + s_2 e^{\xi_2}) - \delta s_1 s_2 e^{\xi_1 + \xi_2}, \tag{3.15b}$$

where $\delta = (p_1 - p_2)^2/(p_1 + p_2)^2$ and $s_i = (1 - p_i)/(1 + p_i)$ $(i = 1, 2)$.

The parametric solution (2.30) represents various types of solutions describing the interaction of two solitons. Here, soliton means one of four types of solutions given in section 3.1. In addition, the solution exhibits a breather solution when the two parameters p_1 and p_2 appear as a complex conjugate pair. Although a number of solutions yield according to the combination of elementary solutions, we present three types of solutions, i.e. kink–kink (or two-kink) solution and kink–loop soliton solution and a breather solution.
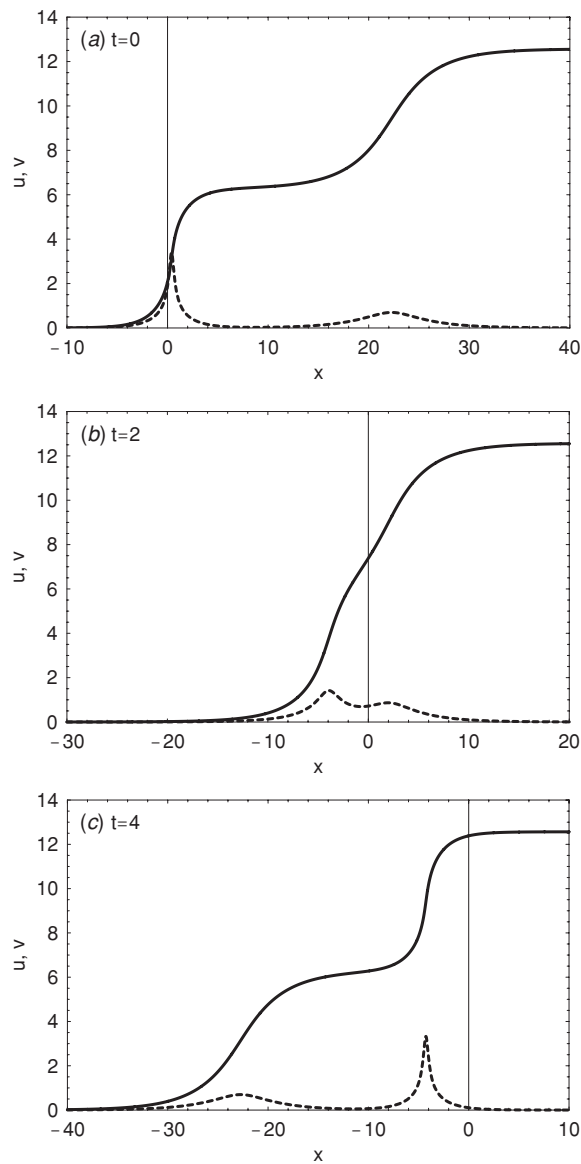
3.2.1. Kink–kink solution. The solid line in figures 5(a)–(c) exhibits the profile of solution u for three different times. The solution represents the 4π-kink. In the same figure, we depict $v \equiv u_x$ by the broken line, showing that it represents the interaction of two solitons. The characteristic of the interaction process of two solitons is seen to be quite different from that of the sG two solitons. Indeed, as evidenced from figure 5, a smaller soliton overtakes, interacts and emerges ahead of a larger soliton. This reflects the fact that the velocity of each soliton is a monotonically decreasing function of its amplitude (see (3.11)). After the interaction, both solitons suffer phase shifts. The general formula for the phase shift arising from the interaction of N solitons will be given by (3.26) below. In particular, for $N = 2$, it reads

$$\Delta_1 = -\frac{1}{p_1} \ln \left(\frac{p_1 - p_2}{p_1 + p_2} \right)^2 - \ln \left(\frac{1 + p_2}{1 - p_2} \right)^2, \tag{3.16a}$$

$$\Delta_2 = \frac{1}{p_2} \ln \left(\frac{p_1 - p_2}{p_1 + p_2} \right)^2 + \ln \left(\frac{1 + p_1}{1 - p_1} \right)^2. \tag{3.16b}$$

In the present example, formula (3.16) yields $\Delta_1 = 4.55$ and $\Delta_2 = -2.42$. A careful inspection of (3.16) reveals that $\Delta_1 > 0$ and $\Delta_2 < 0$ for arbitrary values of p_1 and p_2 satisfying the inequality $0 < p_1 < p_2 < 1/\sqrt{2}$. This implies that the small soliton has moved forward and the large soliton backward relative to the positions they would have reached if both solitons had moved at constant velocities throughout the interaction process. The novel feature of the 2-soliton solution described above will appear here for the first time.

3.2.2. Kink–loop soliton solution. The next example is a solution u representing the interaction of a 2π-kink and a loop soliton. See figures 6(a)–(c). Since the kink propagates to the left and the loop soliton to the right, the solution describes the head-on collision unlike the first example which exhibits an overtaking collision.

Figure 5. The profile of a kink–kink solution u (solid line) and corresponding profile of $v \equiv u_x$ (broken line) for three different times: (a) $t = 0$, (b) $t = 2$, (c) $t = 3$. The parameters are chosen as $p_1 = 0.3$, $p_2 = 0.6$, $\xi_{10} = -5$, $\xi_{20} = 0$.

3.2.3. Breather solution.   The breather solution can be interpreted as a bound state composed of a kink and antikink pair in the sG model [6]. It has a localized structure which oscillates with time and decays exponentially in space. In the generalized sG equation, the similar breather solutions to the sG breathers will be shown to exist. The procedure for constructing breather solutions is the same as that has been used for the short-pulse equation [5]. To be more specific, let

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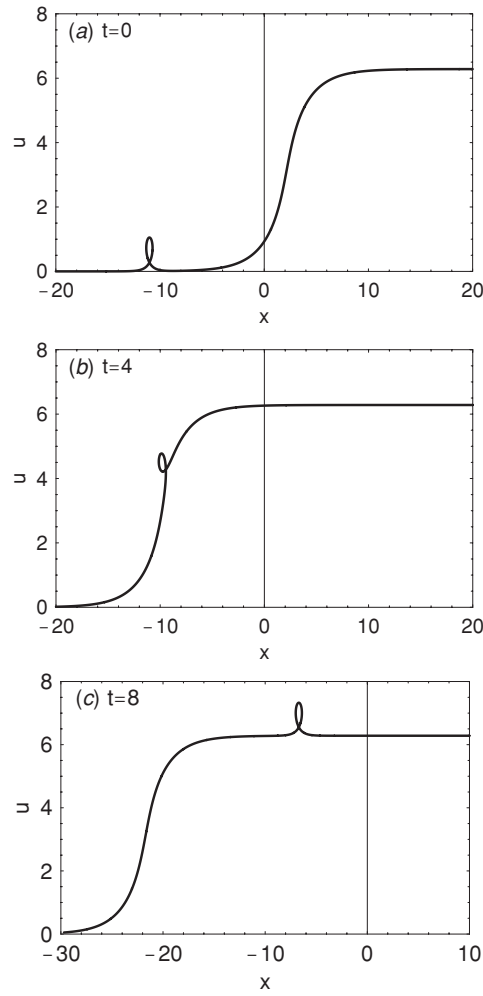
Figure 6. The profile of a kink–loop soliton solution u for three different times: (a) $t = 0$, (b) $t = 4$, (c) $t = 8$. The parameters are chosen as $p_1 = 0.5$, $p_2 = 2.0$, $\xi_{10} = 0$, $\xi_{20} = 25$.

$$p_1 = a + \mathrm{i}b, \qquad p_2 = a - \mathrm{i}b = p_1^*, \qquad (a > 0, b > 0), \tag{3.17a}$$

$$\xi_{10} = \lambda + \mathrm{i}\mu, \qquad \xi_{20} = \lambda - \mathrm{i}\mu = \xi_1^*, \qquad (\lambda, \mu : \text{real}). \tag{3.17b}$$

Then, f and g from (3.5) become

$$f = 1 + \mathrm{i}(\mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_1^*}) + \left(\frac{b}{a}\right)^2 \mathrm{e}^{\xi_1 + \xi_1^*}, \tag{3.18a}$$

$$g = 1 + \mathrm{i}(s_1\,\mathrm{e}^{\xi_1} + s_1^*\,\mathrm{e}^{\xi_1^*}) + s_1 s_1^* \left(\frac{b}{a}\right)^2 \mathrm{e}^{\xi_1 + \xi_1^*}, \tag{3.18b}$$

where

$$\xi_1 = \theta + \mathrm{i}\chi, \tag{3.18c}$$

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$$\theta = a\left(y + \frac{1}{a^2 + b^2}\tau\right) + \lambda, \tag{3.18d}$$

$$\chi = b\left(y - \frac{1}{a^2 + b^2}\tau\right) + \mu, \tag{3.18e}$$

$$s_1 = \frac{1 - a^2 - b^2 - 2\mathrm{i}b}{(1+a)^2 + b^2} \equiv \alpha \mathrm{e}^{-\mathrm{i}\beta}. \tag{3.18f}$$

One can rewrite f and g in terms of the new variables defined by (3.18) as

$$f = 1 + \left(\frac{b}{a}\right)^2 \mathrm{e}^{2\theta} + 2\mathrm{i}\,\mathrm{e}^{\theta}\cos\chi, \tag{3.19a}$$

$$g = 1 + \alpha^2\left(\frac{b}{a}\right)^2 \mathrm{e}^{2\theta} + 2\mathrm{i}\alpha\mathrm{e}^{\theta}\cos(\chi - \beta). \tag{3.19b}$$

The regular solution is obtainable if inequality (2.34) holds with f and g being given by (3.19). An inspection reveals that if a/b is sufficiently small compared to 1, then the solution would exhibit no singularities. However, it is not easy to extract the condition for the regularity from (2.34) when compared with the corresponding problem for the breather solution of the short-pulse equation [5]. We leave it to a future work. Instead, we present a regular solution by a numerical example. In figures 7(a)–(c), the profile of $v \equiv u_x$ is depicted for three different times. We can observe that the breather propagates to the left while changing its profile. The propagation characteristic of the breather is similar to that of the short-pulse equation [5].

3.3. N-soliton solutions

The solutions including an arbitrary number of solitons can be contracted from (2.30) with the tau functions (2.35). There exist a variety of solutions which are composed of any combination of 1-soliton solutions presented in section 3.1. Here, we address the N-kink solutions and $M(=2N)$ breather solutions. For the former solutions, we investigate the asymptotic behavior of solutions for large time and derive the formulas for the phase shift while for the latter ones, we provide a recipe for constructing M breather solution from the N-soliton solution. As examples, we present a solution describing the interaction between a soliton and a breather as well as a two-breather solution.

3.3.1. N-kink solution. Let the velocity of the jth kink be $c_j = (1/p_j^2) - 1$, $(0 < p_j < 1/\sqrt{2})$ and order the magnitude of the velocity of each kink as $c_1 > c_2 > \cdots > c_N$. We observe the interaction of N kinks in a moving frame with a constant velocity c_n. We take the limit $t \to -\infty$ with the phase variable ξ_n being fixed. We then find that f and g have the following leading-order asymptotics

$$f \sim \delta_n \exp\left[\sum_{j=n+1}^{N}\left(\xi_j + \frac{\pi}{2}\mathrm{i}\right)\right]\left(1 + \mathrm{i}\,\mathrm{e}^{\xi_n + \delta_n^{(-)}}\right), \tag{3.20a}$$

$$g \sim \delta_n \exp\left[\sum_{j=n+1}^{N}\left(\xi_j - 2d_j + \frac{\pi}{2}\mathrm{i}\right)\right]\left(1 - \mathrm{i}\,\mathrm{e}^{\xi_n - 2d_n + \delta_n^{(-)}}\right), \tag{3.20b}$$
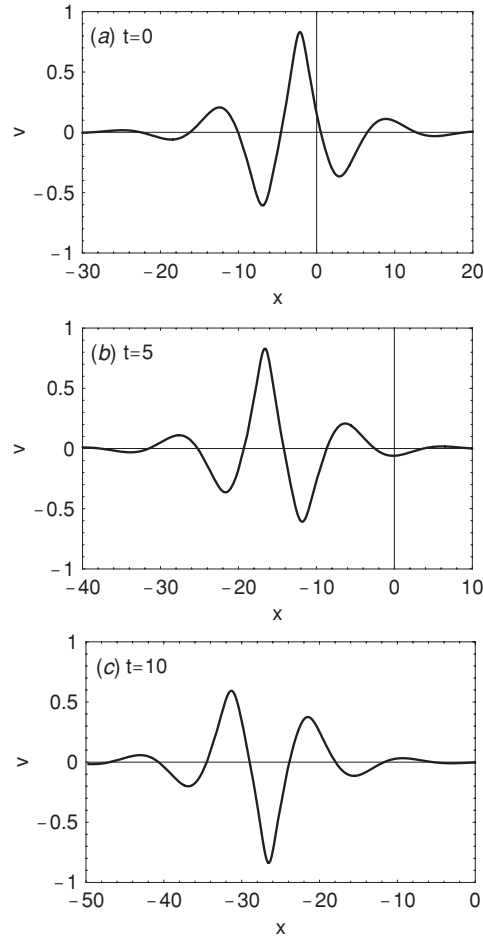
Figure 7. The profile of a breather solution $v \equiv u_x$ for three different times: (a) $t = 0$, (b) $t = 5$, (c) $t = 10$. The parameters are chosen as $p_1 = 0.2 + 0.5\,\mathrm{i}$, $p_2 = p_1^* = 0.2 - 0.5\,\mathrm{i}$, $\xi_{10} = \xi_{20}^* = 0$.

where

$$\delta_n^{(-)} = \sum_{j=n+1}^{N} \ln \left(\frac{p_n - p_j}{p_n + p_j} \right)^2, \tag{3.20c}$$

$$\delta_n = \prod_{n+1 \leqslant j < k \leqslant N} \left(\frac{p_j - p_k}{p_j + p_k} \right)^2. \tag{3.20d}$$

If we substitute (3.20) into (2.30), we obtain the asymptotic form of u and x:

$$u \sim 2 \tan^{-1} \left[\frac{\sinh \left(\xi_n - d_n + \delta_n^{(-)} \right)}{\cosh d_n} \right] + \pi, \tag{3.21a}$$

$$x \sim y + \tau - \ln \frac{1 + p_n \tanh \left(\xi_n - d_n + \delta_n^{(-)} \right)}{1 - p_n \tanh \left(\xi_n - d_n + \delta_n^{(-)} \right)} - 4 \sum_{j=n+1}^{N} d_j - 2d_n + y_0. \tag{3.21b}$$

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Note that we have used equivalent but different expressions for u and x from those given by (3.3).

As $t \to +\infty$, the expressions corresponding to (3.21) are given by

$$u \sim 2 \tan^{-1} \left[\frac{\sinh \left(\xi_n - d_n + \delta_n^{(+)} \right)}{\cosh d_n} \right] + \pi, \tag{3.22a}$$

$$x \sim y + \tau - \ln \frac{1 + p_n \tanh \left(\xi_n - d_n + \delta_n^{(+)} \right)}{1 - p_n \tanh \left(\xi_n - d_n + \delta_n^{(+)} \right)} - 4 \sum_{j=1}^{n-1} d_j - 2d_n + y_0, \tag{3.22b}$$

with

$$\delta_n^{(+)} = \sum_{j=1}^{n-1} \ln \left(\frac{p_n - p_j}{p_n + p_j} \right)^2. \tag{3.22c}$$

Let x_c be the center position of the nth kink in the (x, t) coordinate system. It simply stems from the relations $\xi_n - d_n + \delta_n^{(\pm)} = 0$ by invoking (3.21a) and (3.22a). Thus, as $t \to -\infty$

$$x_c + c_n t + x_{n0} \sim \frac{1}{p_n} \left(d_n - \delta_n^{(-)} \right) - 4 \sum_{j=n+1}^{N} d_j - 2d_n + y_0, \tag{3.23}$$

where $x_{n0} = \xi_{n0}/p_n$. As $t \to +\infty$, on the other hand, the corresponding expression turns out to be

$$x_c + c_n t + x_{n0} \sim \frac{1}{p_n} \left(d_n - \delta_n^{(+)} \right) - 4 \sum_{j=1}^{n-1} d_j - 2d_n + y_0. \tag{3.24}$$

If we take into account the fact that all kinks propagate to the left, we can define the phase shift of the nth kink as

$$\Delta_n = x_c(t \to -\infty) - x_c(t \to +\infty). \tag{3.25}$$

Using (2.36c), (3.20c), (3.22c), (3.23) and (3.24), we find that

$$\Delta_n = \frac{1}{p_n} \left\{ \sum_{j=1}^{n-1} \ln \left(\frac{p_n - p_j}{p_n + p_j} \right)^2 - \sum_{j=n+1}^{N} \ln \left(\frac{p_n - p_j}{p_n + p_j} \right)^2 \right\}$$
$$+ \sum_{j=1}^{n-1} \ln \left(\frac{1 + p_j}{1 - p_j} \right)^2 - \sum_{j=n+1}^{N} \ln \left(\frac{1 + p_j}{1 - p_j} \right)^2 \quad (n = 1, 2, \ldots, N). \tag{3.26}$$

The first term on the right-hand side of (3.26) coincides with the formula for the phase shift arising from the interaction of N kinks of the sG equation [7, 8, 10], whereas the second and third terms appear as a consequence of the coordinate transformation (2.3).

3.3.2. M-breather solution. The construction of the M-breather solution can be done following the similar procedure to that for the 1-breather solution developed in section 3.2.3. Here, $M(=2N)$ is a positive even number. To proceed, we specify the parameters in (2.35) and (2.36) for the tau functions f and g as

$$p_{2j-1} = p_{2j}^* \equiv a_j + \mathrm{i} b_j, \qquad a_j > 0, \quad b_j > 0 \quad (j = 1, 2, \ldots, M), \tag{3.27a}$$

$$\xi_{2j-1,0} = \xi_{2j,0}^* \equiv \lambda_j + \mathrm{i} \mu_j \qquad (j = 1, 2, \ldots, M), \tag{3.27b}$$

where a_j and b_j are positive parameters and λ_j and μ_j are real parameters, respectively. Then, the phase variables ξ_{2j-1} and ξ_{2j} are written as

$$\xi_{2j-1} = \theta_j + i\chi_j \qquad (j = 1, 2, \ldots, M), \tag{3.28a}$$

$$\xi_{2j} = \theta_j - i\chi_j \qquad (j = 1, 2, \ldots, M), \tag{3.28b}$$

with

$$\theta_j = a_j(y + c_j\tau) + \lambda_j \qquad (j = 1, 2, \ldots, M), \tag{3.28c}$$

$$\chi_j = b_j(y - c_j\tau) + \mu_j \qquad (j = 1, 2, \ldots, M), \tag{3.28d}$$

$$c_j = \frac{1}{a_j^2 + b_j^2} \qquad (j = 1, 2, \ldots, M). \tag{3.28e}$$

The parametric solution (2.30) with (3.27) and (3.28) describes multiple collisions of M regular breathers provided that a certain condition is imposed on the parameters a_j and $b_j(j = 1, 2, \ldots, M)$. Although it will be a difficult task to derive the condition for the regularity through inequality (2.34), numerical examples confirm the existence of regular multibreather solutions. See also an example of the 2-breather solution of the short-pulse equation [5]. The asymptotic analysis for the M-breather solution can be performed following the procedure for the N-kink solution, showing that the M-breather solution splits into M single breathers as $t \to \pm\infty$. The resulting asymptotic form is, however, too complicated to write down and hence we omit the detail. One can refer to the similar analysis to that for the M-breather solution of the short-pulse equation [5].

3.3.3. Soliton-breather solution. We take a 3-soliton solution with parameters p_j and ξ_{0j} $(j = 1, 2, 3)$. If one impose the conditions $p_2 = p_1^*, \xi_{02} = \xi_{01}^*$ as already specified for the breather solution (see section 3.2.3) and $0 < p_3 < 1/\sqrt{2}, \xi_{03}$(:real) for the regular kink solution, then the expression of $v \equiv u_x$ would represent a solution describing the interaction between a soliton and a breather. We choose p_1, p_2, ξ_{10} and ξ_{20} as those given by (3.17). Then, the tau functions f and g from (2.35) become

$$f = 1 + i(e^{\xi_1} + e^{\xi_1^*} + e^{\xi_3}) + \left(\frac{b}{a}\right)^2 e^{\xi_1+\xi_1^*} - \delta_{13}e^{\xi_1+\xi_3} - \delta_{13}^* e^{\xi_1^*+\xi_3} + i\left(\frac{b}{a}\right)^2 \delta_{13}\delta_{13}^* e^{\xi_1+\xi_1^*+\xi_3}, \tag{3.29a}$$

$$g = 1 + i(s_1 e^{\xi_1} + s_1^* e^{\xi_1^*} + s_3 e^{\xi_3}) + \left(\frac{b}{a}\right)^2 s_1 s_1^* e^{\xi_1+\xi_1^*} - \delta_{13}s_1 s_3 e^{\xi_1+\xi_3} - \delta_{13}^* s_1^* s_3 e^{\xi_1^*+\xi_3}$$

$$+ i\left(\frac{b}{a}\right)^2 \delta_{13}\delta_{13}^* s_1 s_1^* s_3 e^{\xi_1+\xi_1^*+\xi_3}, \tag{3.29b}$$

where

$$s_1 = \frac{1 - a - ib}{1 + a + ib} = s_2^*, \qquad s_3 = \frac{1 - p_3}{1 + p_3}, \qquad \delta_{13} = \left(\frac{a - p_3 + ib}{a + p_3 + ib}\right)^2 = \delta_{23}^*. \tag{3.29c}$$

Figures 8(a)–(c) show a profile of $v \equiv u_x$ for three different times. We see that as time evolves the soliton overtakes the breather whereby it suffers a phase shift. An asymptotic analysis using the tau functions (3.29) leads to the formula for the phase shift of the soliton, which we denote as Δ. Actually, one has for $p_3^2 < a^2 + b^2$

$$\Delta = \frac{2}{p_3} \ln \frac{(p_3 + a)^2 + b^2}{(p_3 - a)^2 + b^2} - 2\ln \frac{(1 + a)^2 + b^2}{(1 - a)^2 + b^2} \tag{3.30a}$$
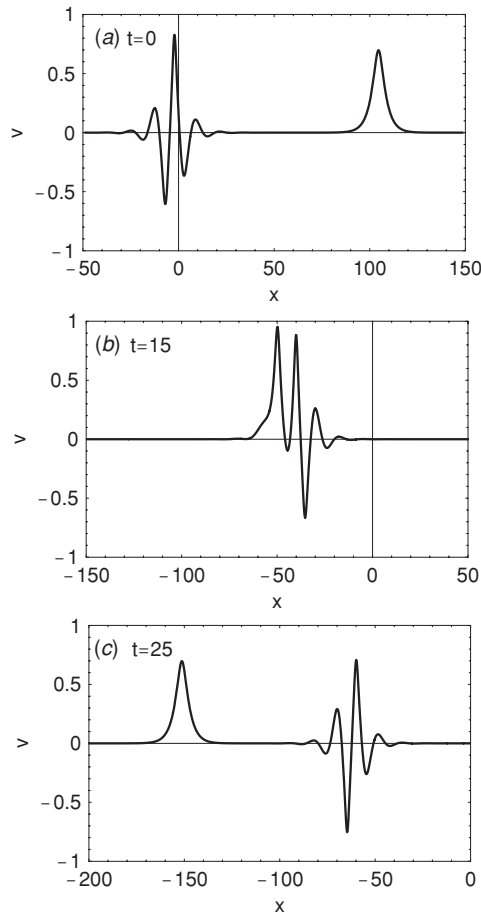
Figure 8. The profile of a soliton–breather solution $v \equiv u_x$ for three different times: (*a*) $t = 0$, (*b*) $t = 15$, (*c*) $t = 25$. The parameters are chosen as $p_1 = 0.2 + 0.5\,\mathrm{i}$, $p_2 = p_1^* = 0.2 - 0.5\,\mathrm{i}$, $p_3 = 0.3$, $\xi_{10} = \xi_{20} = 0$, $\xi_{30} = -30$.

and for $a^2 + b^2 < p_3^2$

$$\Delta = -\frac{2}{p_3} \ln \frac{(p_3 + a)^2 + b^2}{(p_3 - a)^2 + b^2} + 2 \ln \frac{(1 + a)^2 + b^2}{(1 - a)^2 + b^2}. \tag{3.30b}$$

In the present example, formula (3.30*a*) gives $\Delta = 3.07$.

3.3.4. Breather–breather solution.   The breather–breather (or 2-breather) solution is reduced from a 4-soliton solution following the procedure described in section 3.3.2. Figures 9(*a*)–(*c*) show a profile of $v \equiv u_x$ for three different times. It represents a typical feature common to the interaction of solitons, i.e. each breather recovers its profile after collision.

4. Reduction to the short-pulse and sG equations

The short-pulse equation was proposed as a model nonlinear equation describing the propagation of ultra-short optical pulses in nonlinear media [11]. It may be written in an
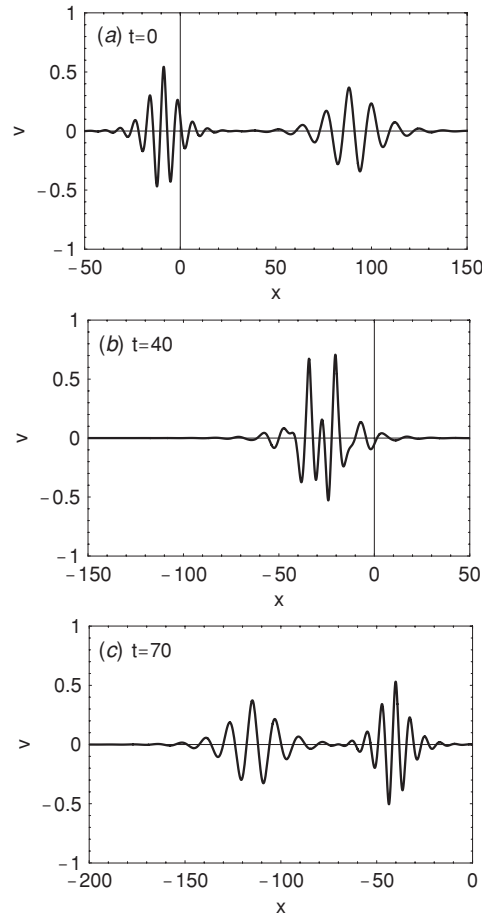
Figure 9. The profile of a breather–breather solution $v \equiv u_x$ for three different times: (*a*) $t = 0$, (*b*) $t = 40$, (*c*) $t = 70$. The parameters are chosen as $p_1 = 0.1 + 0.5\,\mathrm{i}$, $p_2 = 0.16 + 0.8\,\mathrm{i}$, $p_3 = p_1^* = 0.1 - 0.5\,\mathrm{i}$, $p_4 = p_2^* = 0.16 - 0.8\,\mathrm{i}$, $\xi_{10} = \xi_{30}^* = -10$, $\xi_{20} = \xi_{40}^* = 0$.

appropriate dimensionless form as

$$u_{tx} = u + \tfrac{1}{6}(u^3)_{xx}, \tag{4.1}$$

where $u = u(x, t)$ represents the magnitude of the electric field. Here, we demonstrate that the generalized sG equation is reduced to the short-pulse equation by taking an appropriate scaling limit combined with a coordinate transformation. The *N*-soliton solution of the short-pulse equation as well as the formula of the phase shift can be derived from those of the generalized sG equation. The reduction to the SG equation is shown to be established as well by another scaling limit.

4.1. Reduction to the short-pulse equation

Let us first introduce new variable $\bar{u}, \bar{x}$ and $\bar{t}$ according to the relations

$$\bar{u} = \frac{u}{\epsilon}, \qquad \bar{x} = \frac{1}{\epsilon}(x - t), \qquad \bar{t} = \epsilon t, \tag{4.2}$$

where ϵ is a small parameter and the quantities with bar are assumed to be order 1. Rewriting the derivatives in terms of the new variables $\bar{t}$ and $\bar{x}$ and expanding $\sin \epsilon \bar{u}$ in an infinite series with respect to ϵ, we can develop equation (1.1) with $\nu = -1$ to

$$\epsilon \left(\bar{u}_{\bar{t}\bar{x}} - \frac{1}{\epsilon^2} \bar{u}_{\bar{x}\bar{x}} \right) = \epsilon \bar{u} - \frac{\epsilon^3}{6} \bar{u}^3 + \cdots - \frac{1}{\epsilon^2} \partial_{\bar{x}}^2 \left(\epsilon \bar{u} - \frac{\epsilon^3}{6} \bar{u}^3 + \frac{\epsilon^5}{120} \bar{u}^5 + \cdots \right). \tag{4.3a}$$

Note that the terms of order ϵ^{-1} are canceled. Thus, we have

$$\epsilon \bar{u}_{\bar{t}\bar{x}} = \epsilon \left(\bar{u} + \tfrac{1}{6} (\bar{u}^3)_{\bar{x}\bar{x}} \right) + O(\epsilon^3). \tag{4.3b}$$

If we divide both sides of (4.3b) by ϵ and then take the limit $\epsilon \to 0$, we arrive at the short-pulse equation (4.1) written by the new variables. It is noteworthy that the key relation (2.1) has been used to transform the short-pulse equation into the sG equation and it is invariant under the scaling (4.2). The similar scaling variables to (4.2) have been used to derive the short-wave models of the Camassa–Holm and Degasperis–Procesi equations [12].

4.1.1. Scaling limit of the N-soliton solution.   In order to perform the scaling limit of the *N*-soliton solution, we find it appropriate to employ the following new variables in addition to the variables defined by (4.2):

$$\bar{y} = \frac{y}{\epsilon} \qquad \bar{y}_0 = \frac{y_0}{\epsilon}, \qquad \bar{\tau} = \epsilon \tau, \qquad \bar{p}_j = \epsilon p_j, \qquad \bar{\xi}_{j0} = \xi_{j0} \quad (j = 1, 2, \ldots, N). \tag{4.4}$$

If one rewrites the tau function f from (2.35a) in terms of these variables, one simply has

$$f = \bar{f} \equiv \sum_{\mu=0,1} \exp \left[\sum_{j=1}^{N} \mu_j \left(\bar{\xi}_j + \frac{\pi}{2} \mathrm{i} \right) + \sum_{1 \leqslant j < k \leqslant N} \mu_j \mu_k \bar{\gamma}_{jk} \right], \tag{4.5a}$$

with

$$\bar{\xi}_j = \bar{p}_j \bar{y} + \frac{\bar{\tau}}{\bar{p}_j} + \bar{\xi}_{j0} \qquad (j = 1, 2, \ldots, N), \tag{4.5b}$$

$$\mathrm{e}^{\bar{\gamma}_{jk}} = \left(\frac{\bar{p}_j - \bar{p}_k}{\bar{p}_j + \bar{p}_k} \right)^2 \qquad (j, k = 1, 2, \ldots, N; \, j \neq k). \tag{4.5c}$$

In proceeding to the limiting procedure for the tau function g, we need to retain terms up to order ϵ. Thus, the expansion

$$\exp \left(-2 \sum_{j=1}^{N} \mu_j d_j \right) = \prod_{j=1}^{N} \left(\frac{1 - p_j}{1 + p_j} \right)^{\mu_j} \sim \exp \left(-\pi \mathrm{i} \sum_{j=1}^{N} \mu_j \right) \left(1 - 2\epsilon \sum_{j=1}^{N} \frac{\mu_j}{\bar{p}_j} \right) + O(\epsilon^2), \tag{4.6}$$

as well as the scaled variables (4.2) and (4.4), is substituted into (2.35b) to derive the expansion of g:

$$g \sim \sum_{\mu=0,1} \left(1 - 2\epsilon \sum_{j=1}^{N} \frac{\mu_j}{\bar{p}_j} \right) \exp \left[\sum_{j=1}^{N} \mu_j \left(\bar{\xi}_j - \frac{\pi}{2} \mathrm{i} \right) + \sum_{1 \leqslant j < k \leqslant N} \mu_j \mu_k \bar{\gamma}_{jk} \right] + O(\epsilon^2)$$

$$= \bar{f}^* - 2\epsilon \bar{f}_{\bar{\tau}}^* + O(\epsilon^2). \tag{4.7}$$

Insertion of (4.2), (4.5) and (4.7) into (2.30a) gives

$$\epsilon \bar{u} = \mathrm{i} \ln \frac{\bar{f}^*(\bar{f} - 2\epsilon \bar{f}_\tau)}{\bar{f}(\bar{f}^* - 2\epsilon \bar{f}_\tau^*)} + O(\epsilon^2). \tag{4.8}$$

In the limit of $\epsilon \to 0$, (4.8) leads to the scaling limit of u

$$\bar{u} = 2\mathrm{i} \left(\ln \frac{\bar{f}^*}{\bar{f}} \right)_{\bar{\tau}}. \tag{4.9}$$

Applying the similar procedure to (2.30b), we find

$$\epsilon \bar{x} = \epsilon \bar{y} - 2\epsilon (\ln \bar{f}^* \bar{f})_{\bar{\tau}} + \epsilon \bar{y}_0 + O(\epsilon^2). \tag{4.10}$$

It follows from (4.10) that

$$\bar{x} = \bar{y} - 2(\ln \bar{f}^* \bar{f})_{\bar{\tau}} + \bar{y}_0. \tag{4.11}$$

Expressions (4.9) and (4.11) coincide with the parametric representation of the N-soliton solution of the short-pulse equation [5].

4.1.2. Scaling limit of the phase shift.   The scaling limit of formula (3.26) for the phase shift can be derived easily. Indeed, if we define the new variable for the phase shift by $\bar{\Delta}_n = \Delta_n/\epsilon$ and substitute this expression and the scaled variable $\bar{p}_j = \epsilon p_j$ from (4.4) into (3.26), we find, after taking the limit $\epsilon \to 0$, that

$$\bar{\Delta}_n = \frac{1}{\bar{p}_n} \left\{ \sum_{j=1}^{n-1} \ln \left(\frac{\bar{p}_n - \bar{p}_j}{\bar{p}_n + \bar{p}_j} \right)^2 - \sum_{j=n+1}^{N} \ln \left(\frac{\bar{p}_n - \bar{p}_j}{\bar{p}_n + \bar{p}_j} \right)^2 \right\}$$
$$+ \sum_{j=1}^{n-1} \frac{4}{\bar{p}_j} - \sum_{j=n+1}^{N} \frac{4}{\bar{p}_j}, \qquad (n = 1, 2, \ldots, N). \tag{4.12}$$

This expression is just the corresponding formula for the short-pulse equation [5]. Note that if all $\bar{p}_j$ are real parameters such that $\bar{p}_j \neq \bar{p}_k$ for $j \neq k$, then (4.12) gives the formula for the phase shift resulting from the overtaking collisions of N loop solitons.

4.2. Reduction to the sG equation

The reduction to the sG equation is rather straightforward compared to the previous one for the short-pulse equation. It turns out that the appropriate scaled variables are given by

$$\bar{u} = u, \qquad \bar{x} = \epsilon x, \qquad \bar{y} = \epsilon y, \qquad \bar{t} = \frac{t}{\epsilon}, \qquad \bar{\tau} = \frac{\tau}{\epsilon},$$
$$\bar{p}_j = \frac{p_j}{\epsilon}, \qquad \bar{\xi}_{j0} = \xi_{j0} \quad (j = 1, 2, \ldots, N). \tag{4.13}$$

In terms of the variables (4.13), we can recast equation (1.1) to the sG equation $\bar{u}_{\bar{t}\bar{x}} = \sin \bar{u}$ in the limit of $\epsilon \to 0$. The parametric solution (2.30) reduces to the usual form of the N-soliton solution of the sG equation, i.e. $\bar{u}(\bar{x}, \bar{t}) = 2\mathrm{i} \ln(\bar{f}^*/\bar{f})$ where $\bar{f}$ is given by (4.5) with the identification $\bar{y} = \bar{x}$, $\bar{\tau} = \bar{t}$. The phase shift is scaled by $\bar{\Delta}_n = \epsilon \Delta_n$. It is given by the first term on the right-hand side of (4.12), reproducing the well-known formula derived by the asymptotic analysis of the N-soliton solution of the sG equation [7, 8, 10].

5. Conclusion

A direct approach employed in this paper constructs various types of soliton solutions such as single- and multi-valued kinks, loop solitons and breathers. These elementary solutions are combined to produce a variety of multisoliton solutions. As examples, we presented a solution describing the interaction between a soliton and a breather as well as a 2-breather solution. As far as solutions are concerned, one can observe that the generalized sG equation has a rich structure compared with that of the sG equation. We also demonstrated that the generalized sG equation is reduced to both the short-pulse and sG equations in appropriate scaling limits. Another interesting issue is the generalized sG equation (1.1) with $\nu = 1$ for which the method of solution still remains open. One may apply a sequence of nonlinear transformation similar to that used here to obtain solutions. Another direction to be worth investigating is the periodic problem. The exact method of solution used here will work well for constructing periodic solutions of equation (1.1). See [13, 14] for periodic solutions of the short-pulse equation. These problems will be pursued in future works.

Note added in proof. After the acceptance of this paper for publication, the author noticed that equations (2.10) and (2.11) have been initially derived by M D Kruskal [15]. The author thanks Dr Maxim Pavlov for bringing this useful information to me.

Appendix A. Proof of theorem 2.2

In this appendix, we show that the tau functions f and g given, respectively, by (2.35a) and (2.35b) satisfy the bilinear equations (2.24). First, we rewrite f and g in terms of determinants. For this purpose, we use the formula [16]

$$\sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \xi_j + \sum_{1\leqslant j<k\leqslant N} \mu_j \mu_k \gamma_{jk}\right] = \lambda_N \det\left(e^{\zeta_j}\delta_{jk} + \frac{2p_j}{p_j + p_k}\right)_{1\leqslant j,k\leqslant N}, \qquad (A.1)$$

where

$$\zeta_j = \xi_j + \sum_{\substack{k=1 \\ (k\neq j)}}^{N} \gamma_{jk}, \qquad \lambda_N = \exp\left(-\sum_{1\leqslant j<k\leqslant N} \gamma_{jk}\right), \qquad (A.2)$$

and δ_{jk} is Kronecker's delta. Since numerical factors multiplied by f and g have no effects on the proof of (2.24), we use the determinantal expression given on the right hand of (A.1) instead of the finite sum. Furthermore, we shift the phase factor ξ_{j0} by $-\sum_{\substack{k=1 \\ (k\neq j)}}^{N} \gamma_{jk}$ so that $\zeta_j = \xi_j$. Consequently, we can express f and g by the following determinants:

$$f = \det A \equiv |A|, \qquad A = (a_{jk})_{1\leqslant j,k\leqslant N}, \qquad a_{jk} = \mathrm{i}\,e^{\xi_j}\delta_{jk} + \frac{2p_j}{p_j + p_k}, \qquad (A.3)$$

$$g = \det B \equiv |B|, \qquad B = (b_{jk})_{1\leqslant j,k\leqslant N}, \qquad b_{jk} = \mathrm{i}\frac{1 - p_j}{1 + p_j}e^{\xi_j}\delta_{jk} + \frac{2p_j}{p_j + p_k}. \qquad (A.4)$$

For later convenience, we introduce some notations as well as formulas for determinants. Matrices and cofactors associated with any $N \times N$ matrix $A = (a_{jk})_{1\leqslant j,k\leqslant N}$ are defined as follows:

$$A(\mathbf{a}; \mathbf{b}) = \begin{pmatrix} a_{11} & \ldots & a_{1N} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{N1} & \ldots & a_{NN} & b_N \\ a_1 & \ldots & a_N & 0 \end{pmatrix}, \tag{A.5}$$

$$A(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}) = \begin{pmatrix} a_{11} & \ldots & a_{1N} & c_1 & d_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{N1} & \ldots & a_{NN} & c_N & d_N \\ a_1 & \ldots & a_N & 0 & 0 \\ b_1 & \ldots & b_N & 0 & 0 \end{pmatrix}, \tag{A.6}$$

$$A_{jk} = \frac{\partial |A|}{\partial a_{jk}}, \tag{A.7}$$

Here, A_{jk} is the cofactor of a_{jk} and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are N-dimensional vectors, $\mathbf{a} = (a_1, a_2, \ldots, a_N)$ for example. The following formulas are used frequently in the present analysis [17]:

$$\begin{vmatrix} a_{11} & \ldots & a_{1N} & x_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{N1} & \ldots & a_{NN} & x_N \\ y_1 & \ldots & y_N & z \end{vmatrix} = |A|z - \sum_{j,k=1}^{N} A_{jk} x_j y_k, \tag{A.8}$$

$$|A| = \begin{vmatrix} a_{11} - 1 & \ldots & a_{1N} - 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{N1} - 1 & \ldots & a_{NN} - 1 & 1 \\ -1 & \ldots & -1 & 1 \end{vmatrix}, \tag{A.9}$$

$$|A(\mathbf{a} + \mathbf{b}; \mathbf{c} + \mathbf{d})| = |A(\mathbf{a}; \mathbf{c})| + |A(\mathbf{a}; \mathbf{d})| + |A(\mathbf{b}; \mathbf{c})| + |A(\mathbf{b}; \mathbf{d})|, \tag{A.10}$$

$$|A(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})||A| = |A(\mathbf{a}; \mathbf{c})||A(\mathbf{b}; \mathbf{d})| - |A(\mathbf{a}; \mathbf{d})||A(\mathbf{b}; \mathbf{c})|, \tag{A.11}$$

$$\sum_{j,k=1}^{N} (f_j + g_k) a_{jk} A_{jk} = \sum_{j=1}^{N} (f_j + g_j)|A|. \tag{A.12}$$

Formula (A.11) is Jacobi's identity and formula (A.12) follows from the expansion formulas for determinants $\sum_{k=1}^{N} a_{ik} A_{jk} = \delta_{ij}|A|$, $\sum_{k=1}^{N} a_{ki} A_{kj} = \delta_{ij}|A|$.

Let us now proceed to the proof. First, we modify the determinant $|B|$. We extract a factor $2p_j$ from the jth row of $|B|$ and then extract a factor $(1 + p_j)^{-1}$ from the jth column ($j = 1, 2, \ldots, N$). Subsequently, the determinant is modified by formula (A.9). We extract a factor $1 - p_j$ from the jth row of the resultant determinant and then multiply the jth row by a factor $2p_j$ ($j = 1, 2, \ldots, N$). We then find

$$g = \mu(|A| + 2|A(-\mathbf{1}; \mathbf{q} - \mathbf{1})|) = \mu(|A| + 2|A(\mathbf{1}; \mathbf{1})| - 2|A(\mathbf{1}; \mathbf{q})|), \tag{A.13}$$

where

$$\mathbf{q} = \left(\frac{1}{1 - p_1}, \frac{1}{1 - p_2}, \ldots, \frac{1}{1 - p_N} \right), \quad \mathbf{1} = (1, 1, \ldots, 1), \quad \mu = \frac{\prod_{j=1}^{N}(1 - p_j)}{\prod_{j=1}^{N}(1 + p_j)}. \tag{A.14}$$

The last line of (A.13) is a consequence of formula (A.10). If we use (A.9), we can rewrite (A.13) as

$$g = \mu(|\bar{A}| + |\bar{A}(\mathbf{1}; \mathbf{1})| - 2|\bar{A}(\mathbf{1}; \mathbf{q})|), \tag{A.15}$$

where $\bar{A}$ is a skew-Hermitian matrix defined by

$$\bar{A} = (\bar{a}_{jk})_{1 \leqslant j,k \leqslant N}, \qquad \bar{a}_{jk} = \mathrm{i}\,\mathrm{e}^{\xi_j}\delta_{jk} + \frac{p_j - p_k}{p_j + p_k}. \tag{A.16}$$

Since $|\bar{A}|^* = (-1)^N|\bar{A}|$, the complex conjugate of g becomes

$$g^* = (-1)^N\mu(|\bar{A}| - |\bar{A}(\mathbf{1}; \mathbf{1})| + 2|\bar{A}(\mathbf{q}; \mathbf{1})|). \tag{A.17}$$

In view of the formulas $|A| = |\bar{A}| - |\bar{A}(\mathbf{1}; \mathbf{1})|$ and $|\bar{A}(\mathbf{q}; \mathbf{1})| = |A(\mathbf{q}; \mathbf{1})|$ which follow from (A.9), (A.17) reduces to

$$g^* = (-1)^N\mu(|A| + 2|A(\mathbf{q}; \mathbf{1})|). \tag{A.18}$$

Similarly, one has

$$f^* = (-1)^N(|A| + 2|A(\mathbf{1}; \mathbf{1})|). \tag{A.19}$$

It follows from (A.13), (A.18) and (A.19) that

$$\begin{aligned}\tfrac{1}{2}(fg^* - f^*g) = (-1)^N\mu[(|A(\mathbf{1}; \mathbf{q})| + |A(\mathbf{q}; \mathbf{1})|)|A| \\ + 2(|A(\mathbf{1}; \mathbf{q})| - |A(\mathbf{1}; \mathbf{1})|)|A(\mathbf{1}; \mathbf{1})| - 2|A||A(\mathbf{1}; \mathbf{1})|].\end{aligned} \tag{A.20}$$

The next step is to calculate the right-hand side of (2.24a). First, applying the differential rule of determinant to f, one has

$$f_y = \sum_{j,k=1}^{N} \frac{\partial a_{jk}}{\partial y} A_{jk} = \sum_{j,k=1}^{N} (p_j \mathrm{i}\,\mathrm{e}^{\xi_j}\delta_{jk})A_{jk} = \frac{1}{2}\sum_{j,k=1}^{N}(p_j + p_k)\left(a_{jk} - \frac{2p_j}{p_j + p_k}\right)A_{jk}. \tag{A.21}$$

By virtue of (A.8) and (A.12), we can recast (A.21) to

$$f_y = \sum_{j=1}^{N} p_j|A| + |A(\mathbf{1}; \mathbf{p})|, \tag{A.22}$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_N)$. A similar calculation leads to

$$g_y^* = \sum_{j=1}^{N} p_j|B|^* + (-1)^N\mu(|A(\mathbf{1}; \mathbf{1})| + |A(\mathbf{1}; \mathbf{p})| - |A(\mathbf{q}; \mathbf{1})| - |A(\mathbf{q}; \mathbf{p})| - 2|A(\mathbf{q}, \mathbf{1}; \mathbf{p}, \mathbf{1})|). \tag{A.23}$$

It follows from (A.18), (A.22) and (A. 23) that

$$\begin{aligned}D_y g^* \cdot f = (-1)^N\mu[(|A(\mathbf{1}; \mathbf{1})| - |A(\mathbf{q}; \mathbf{1})| - |A(\mathbf{q}; \mathbf{p})| \\ - 2|A(\mathbf{q}, \mathbf{1}; \mathbf{p}, \mathbf{1})|)|A| - 2|A(\mathbf{1}; \mathbf{p})||A(\mathbf{q}; \mathbf{1})|].\end{aligned} \tag{A.24}$$

Using Jacobi's identity (A.11) with $\mathbf{a} = \mathbf{q}$, $\mathbf{b} = \mathbf{1}$, $\mathbf{c} = \mathbf{p}$, $\mathbf{d} = \mathbf{1}$, (A.24) simplifies to

$$D_y g^* \cdot f = (-1)^N\mu[(|A(\mathbf{1}; \mathbf{1})| - |A(\mathbf{q}; \mathbf{1})| - |A(\mathbf{q}; \mathbf{p})|)|A| - 2|A(\mathbf{q}; \mathbf{p})||A(\mathbf{1}; \mathbf{1})|]. \tag{A.25}$$

Referring to (A.20) and (A.25), we obtain

$$\begin{aligned}D_y g^* \cdot f - \tfrac{1}{2}(fg^* - f^*g) \\ = (-1)^N\mu(-|A(\mathbf{1}; \mathbf{1})| + |A(\mathbf{1}; \mathbf{q}))| - |A(\mathbf{q}; \mathbf{p})|)(|A| + 2|A(\mathbf{1}; \mathbf{1})|). \end{aligned} \tag{A.26}$$

Let $P = -|A(\mathbf{1}; \mathbf{1})| + |A(\mathbf{1}; \mathbf{q})| - |A(\mathbf{q}; \mathbf{p})|$. Applying (A.8) with $q_j = 1/(1 - p_j)$, P becomes

$$P = \sum_{j,k=1}^{N} (1 - q_j + p_j q_k) A_{jk} = \sum_{j,k=1}^{N} \frac{p_j(p_k - p_j)}{(1 - p_j)(1 - p_k)} A_{jk}. \tag{A.27}$$

If we extract a factor p_l from the lth row of A_{jk} $(l = 1, 2, \ldots, N; l \neq j)$, P is modified as

$$P = \left(\prod_{l=1}^{N} p_l \right) \sum_{j,k=1}^{N} \frac{p_k - p_j}{(1 - p_j)(1 - p_k)} \hat{A}_{jk}, \tag{A.28}$$

where $\hat{A}_{jk}$ is the cofactor of the (j, k) element of the matrix $\hat{A}$ defined by

$$\hat{A} = (\hat{a}_{jk})_{1 \leqslant j,k \leqslant N}, \qquad \hat{a}_{jk} = \frac{\mathrm{i}e^{\xi_j}}{p_j} \delta_{jk} + \frac{2}{p_j + p_k}. \tag{A.29}$$

Since $\hat{A}$ is a symmetric matrix, $\hat{A}_{jk} = \hat{A}_{kj}$. Taking this relation into (A.28), we conclude that $P = 0$. This completes the proof of (2.24a).

The proof of (2.24b) follows immediately from that of (2.24a) by a symmetry of the tau functions. Indeed, if we exchange the variables y and τ and subsequently replace the parameter p_j by p_j^{-1} $(j = 1, 2, \ldots, N)$, we then see that f from (A.3) is unchanged whereas g from (A.4) is transformed to g^*. Thus, under this manipulation, the bilinear equation (2.24b) turns out to the bilinear equation (2.24a), which completes the proof of (2.24b).

Appendix B. An alternative derivation of the 1-soliton solutions

The 1-soliton solutions take the form of traveling wave:

$$u = u(X), \qquad X = x + c_1 t + x_0. \tag{B.1}$$

Substituting this expression into equation (1.1) with $\nu = -1$ and integrating the resultant ordinary differential equation once with respect to X under the boundary condition $u(-\infty) = 0 \pmod{2\pi}$, we obtain

$$u_X^2 = \frac{(c_1 + 1)^2 - (\cos u + c_1)^2}{(\cos u + c_1)^2}. \tag{B.2}$$

Since $u_X^2 \geqslant 0$, we must require that the right-hand side of (B.2) is nonnegative which imposes the condition on possible values of c_1. One can see that this condition becomes $c_1 \geqslant -\cos^2(u/2)$. To proceed, we define a new variable ξ by

$$X = \int (\cos u + c_1) \, \mathrm{d}\xi. \tag{B.3}$$

Then, equation (B.2) reduces to

$$u_\xi = \pm \sqrt{(c_1 + 1)^2 - (\cos u + c_1)^2}. \tag{B.4}$$

In accordance with values of c_1, the solutions are classified to several types, which we shall now discuss in detail.

(1) $c_1 > 0$. In the case of $c_1 > 0$, (B.4) is integrated through the change of the variable by $s = \tan(u/2)$. After an elementary calculation, we obtain

$$s = \pm \sqrt{\frac{c_1 + 1}{c_1}} \frac{1}{\sinh \sqrt{c_1 + 1}\, \xi} \tag{B.5}$$

and

$$\cos u = \frac{1 - s^2}{1 + s^2} = 1 - \frac{2(c_1 + 1)}{c_1 \sinh^2 \sqrt{c_1 + 1}\,\xi + c_1 + 1}. \tag{B.6}$$

Substituting (B.6) into (B.3) and performing the integration with respect to ξ, we find

$$X = (c_1 + 1)\xi - \ln \frac{\sqrt{c_1 + 1} + \tanh \sqrt{c_1 + 1}\,\xi}{\sqrt{c_1 + 1} - \tanh \sqrt{c_1 + 1}\,\xi} + \xi_0, \tag{B.7}$$

where ξ_0 is an integration constant. It follows from (B.5) and the boundary condition for u that

$$u = 2\tan^{-1}\left(\sqrt{\frac{c_1}{c_1 + 1}}\,\sinh \sqrt{c_1 + 1}\,\xi\right) + \pi. \tag{B.8}$$

If we put

$$c_1 = \frac{1}{p_1^2} - 1 \,(0 < p_1 < 1), \qquad \xi = p_1(\xi_1 - d_1), \qquad d_1 = \tanh^{-1} p_1,$$

$$\xi_0 = \ln\left(\frac{1 - p_1}{1 + p_1}\right) + \frac{d_1}{p_1} + y_0, \tag{B.9}$$

we can see that (B.7) and (B.8) coincide with (3.4) and (3.3*a*), respectively.

(2) $-1 < c_1 < 0$. In this case, a calculation similar to case 1 gives

$$s = \pm\sqrt{\frac{c_1 + 1}{-c_1}}\,\frac{1}{\cosh \sqrt{c_1 + 1}\,\xi} \tag{B.10}$$

and

$$u = -2\tan^{-1}\left(\sqrt{\frac{-c_1}{c_1 + 1}}\,\cosh \sqrt{c_1 + 1}\,\xi\right) + \pi, \tag{B.11}$$

$$X = (c_1 + 1)\xi - \ln \frac{1 + \sqrt{c_1 + 1}\,\tanh \sqrt{c_1 + 1}\,\xi}{1 - \sqrt{c_1 + 1}\,\tanh \sqrt{c_1 + 1}\,\xi} + \xi_0. \tag{B.12}$$

If we put

$$c_1 = \frac{1}{p_1^2} - 1 \,(p_1 > 1), \qquad \xi = p_1(\xi_1 - d_1), \qquad d_1 = \tanh^{-1} \frac{1}{p_1},$$

$$\xi_0 = \ln\left(\frac{p_1 - 1}{p_1 + 1}\right) + \frac{d_1}{p_1} + y_0, \tag{B.13}$$

we can reproduce the parametric solution (3.3*a*) and (3.4).

(3) $c_1 = 0$. For the special value $c_1 = 0$, integration of (B.3) and (B.4) can be performed readily, giving rise to the solution

$$u = \frac{\pi}{2} + \tan^{-1}(\sinh \xi), \tag{B.14}$$

$$X = -\ln(\cosh \xi) + \xi_0. \tag{B.15}$$

It is easy to confirm that (B.14) and (B.15) coincide with (3.12*a*) and (3.12*b*), respectively.

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